Quantum integrable Toda like systems

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Abstract

Using deformation quantization and suitable 2 by 2 quantum R-matrices we show that a list of Toda like classical integrable systems given by Y.B.Suris is quantum integrable in the sense that the classical conserved quantities (which are already in involution with respect to the Poisson bracket) commute with respect to the standard star-product of Weyl type in flat $2n$-dimensional space.

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1 Introduction

During the past decades a lot of new families of systems of Hamiltonian mechanics have been found which are integrable in the sense of Liouville, i.e. which allow for $n = \frac{1}{2}\dim(\text{Phasespace})$ independent, Poisson-commuting integrals of motion on which the Hamiltonian functionally depends: among these are the Calogero-Moser systems and the Toda chains (see e.g. [12], [11]). These systems allow for a formulation in terms of a Lax pair (which proves that the coefficients of the characteristic polynomial and/or the trace polynomials of the matrix Lie algebra valued function L on phase space Poisson-commute with the Hamiltonian) and the existence of a so-called classical r-matrix expressing the Poisson bracket of the components of L in terms of Lie commutators guarantees that the above invariant functions Poisson-commute (see e.g. [4],[9]).

Already at the time of the discovery of the above-mentioned integrable systems the question of their *quantum integrability* had been considered, i.e. whether one can associate to each of these classical conserved quantities a quantum operator (by means of some ordering prescription) such that these quantum operators commute with the Hamiltonian operator and among each other. In [6] (see also [10] for a much more explicit proof) arguments were given that there was no ordering problem for the corresponding integrals for the Calogero-Moser systems and that the corresponding operators should commute (see also [11] for a similar type of argument for the Toda chain).

Recently Y.B.Suris gave a list of Toda-like systems defined by traces over products of 2×2-matrices which are all classically integrable by means of two types of a constant classical r-matrix depending on spectral parameters [13]. Among his systems are relativistic and discretized versions of the original Toda lattice.

The motivation of this article (see also the second author's thesis [14]) was to check whether all the systems of Suris' list are quantum-integrable. In order to control the possible ordering prescriptions we chose to use the concept of deformation quantization defined in [3] which has now been well-established on every symplectic manifold. The advantage of this method to using operators is the fact that the quantum noncommutative multiplication is formulated directly on the space of classical observables as a deformed pointwise multiplication which makes it easier and more natural to compare with classical computations. On flat $\mathbb{R}^{2n}$ (and more generally on every cotangent bundle, see [5]) there exist differential operator representations of the deformed algebra corresponding to canonical quantization with Weyl ordering prescription. Consequently, quantum commutativity of the classical integrals in terms of star-products can be translated into commuting operators if necessary. Another advantage of deformation quantization is that quantum integrability can be formulated on much more general symplectic manifolds where star-products still exist thanks to the theorem of DeWilde-Lecomte [8] but operator representations are a priori lacking.

We find that all the systems given by Suris are quantum integrable, and our proof uses the quantum R-matrices of the form identity plus a multiple of the classical r-matrix. In particular we re-obtain the quantum integrability of the Toda chain. But, as it turned out there is quantum asymmetry: the analogs of the coefficients of the characteristic polynomial are actually commuting with respect to the star-product whereas the analogs of the trace-polynomials are *not*. One can cure that by adding quantum corrections to the trace polynomials which can be obtained by interpreting the Waring identities (which are polynomial formulas between the two sets of functions) in the deformed algebra.

2 Star products and ordering prescription of standard and Weyl ordered products in $\mathbb{R}^{2n}$

In this section we shall briefly recall the formulas needed for the star-products and their operator representations in flat $\mathbb{R}^{2n}$ (see also [1], [5]). We shall denote the co-ordinates of $\mathbb{R}^{2n}$ by $(\vec{q}, \vec{p})$.

Let $F : \mathbb{R}^{2n} \to \mathbb{C}$ be smooth. The standard ordering prescription assigns to F the formal differential operator series

$$(\rho_S(F)\psi)(\vec{q}) := \sum_{k=0}^{\infty} \frac{1}{k!}(\hbar/i)^k \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k F}{\partial p_{i_1} \cdots \partial p_{i_k}}(\vec{q}, \vec{0}) \frac{\partial^k \psi}{\partial q^{i_1} \cdots \partial q^{i_k}}(\vec{q}) \tag{1}$$

However, real-valued functions do not correspond to symmetric operators (on the dense domain of compactly supported smooth complex functions), but one rather has $\rho_S(F)^\dagger = \rho_S(N^2\bar{F})$ where

$$N := exp(\frac{\hbar}{2i} \sum_{j=1}^{n} \frac{\partial^2}{\partial q^j \partial p_j}). \tag{2}$$

The Weyl ordering prescription

$$\rho_W(F) := \rho_S(NF) \tag{3}$$

has the more physical property $\rho_W(F)^\dagger = \rho_W(\bar{F})$ and exactly corresponds to a total symmetrization of position and momentum operators in polynomial observables. The star-products of standard ordered type, $*_S$ and of Weyl type, $*$, of two smooth complex-valued functions F, G on phase space are defined as follows:

$$(F *_S G)(\vec{q}, \vec{p}) := exp(\frac{\hbar}{i} \sum_{j=1}^{n} \frac{\partial^2}{\partial q'^j \partial p_i}) F(\vec{q}, \vec{p}) G(\vec{q}\,', \vec{p}\,') \Big|_{\vec{q}=\vec{q}', \vec{p}=\vec{p}'} \tag{4}$$

$$(F * G)(\vec{q}, \vec{p}) := exp(\frac{i\hbar}{2} \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial q^j \partial p'_j} - \frac{\partial^2}{\partial q'^j \partial p_j} \right)) F(\vec{q}, \vec{p}) G(\vec{q}', \vec{p}') \Big|_{\vec{q}=\vec{q}', \vec{p}=\vec{p}'}, \tag{5}$$

they satisfy the representation identities

$$\rho_S(F *_S G) = \rho_S(F)\rho_S(G) \quad , \quad \rho_W(F * G) = \rho_W(F)\rho_W(G) \quad , \tag{6}$$

and they are related by N as follows:

$$F * G = N^{-1}((NF) *_S (NG)) \quad . \tag{7}$$

Clearly, the two star-products are associative and have the correct classical limit, i.e. one gets pointwise multiplication at the order $\hbar^0$ and i times the Poisson bracket taking the commutator at the order $\hbar$. For practical purposes it is often easier to compute the star-product of standard-ordered type and to use N to switch to the Weyl type multiplication.

We conclude this section with the general definition of a quantum integrable system:

Definition 2.1 (Quantum integrable system) *A classical completely integrable system with Hamilton function H on a $2n$-dimensional symplectic manifold is said to be quantum integrable, if there exists a star product $*$ and n formal power series $F_i \in C^\infty(M)[[\hbar]]$ which conincide with the classical conserved quantities f_i at order zero in $\hbar$, such that*

i.) $F_i * H - H * F_i = 0$

ii.) $F_i * F_j - F_j * F_i = 0$

for all $1 \leq i, j \leq n$.

Remark: i) Writing down the formal power serie ($1 \leq i \leq n$),

$$F_i = \sum_{k=0} \hbar^k F_i^{(k)}, \qquad F_i^{(0)} \equiv f_i$$

higher terms in $\hbar$ of F_i can be regarded as quantum corrections of the classical conserved quantities f_i. ii) Here, we do not consider deformations of the Hamiltonian.

3 Quantum R-matrices for certain classical r-matrices

The field $\mathbb{K}$ is either equal to $\mathbb{R}$ or to $\mathbb{C}$. Let $L(n,\mathbb{K})$ denote the space of $\mathbb{K}$-valued $(n \times n)$-matrices with standard basis E_{ij}, where $1 \leq i, j \leq n$. To express tensor products with spectral parameters properly let $\mathbb{K}(\lambda_1, \lambda_2, \ldots, \lambda_k)$ denote the field of rational functions over $\mathbb{K}$ in k parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$. ρ will be an abbreviation for $i\hbar/2$. Furthermore, recall the standard tensor notation in the context of R-matrices: for an element $R = \sum_i s_i \otimes t_i \otimes \phi_i(\lambda, \mu)$ in $L(n,\mathbb{K}) \otimes L(n,\mathbb{K}) \otimes \mathbb{K}(\lambda, \mu)$ and for a positive integer $N \geq 2$ and integers a, b s.t. $1 \leq a < b \leq N$ we shall write R_{ab} for $\sum_i \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes s_i \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes t_i \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \phi_i(\lambda_a, \lambda_b)$ (where s_i is at the ath tensor factor and t_i is at the bth tensor factor) regarded as an element in $L(n,\mathbb{K})^{\otimes N} \otimes \mathbb{K}(\lambda_1, \cdots, \lambda_N)$. Note that this last space is an associative algebra with respect to tensor factor wise multiplication. The symbol R_{ba} is equal to the above expression with s_i and t_i and λ_a and λ_b exchanged. We shall frequently use the standard isomorphism

$$\begin{aligned} L(n,\mathbb{K}) \otimes L(n,\mathbb{K}) &\rightarrow L(n^2,\mathbb{K}) \ , \\ a \otimes b &\mapsto \begin{pmatrix} a_{11}b & \ldots & a_{1n}b \\ \vdots & \ddots & \vdots \\ a_{n1}b & \ldots & a_{nn}b \end{pmatrix} \end{aligned} \tag{8}$$

to conveniently express tensor products of matrices.

We consider two particular classical r-matrices with spectral parameter, i.e. elements r, $\tilde{r}$ in $L(2,\mathbb{K}) \otimes L(2,\mathbb{K}) \otimes \mathbb{K}(\lambda, \mu)$ which are antisymmetric ($r_{12} = -r_{21}$) and obey the classical Yang-Baxter equation,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad , \tag{9}$$

which have been used in a preprint by Y.B.Suris [13]. The first one is generated by the so-called Casimir element $C := \sum_{i,j=1}^{2} E_{ij} \otimes E_{ji}$

$$r := \frac{C}{\lambda - \mu} \quad , \tag{10}$$

the second one is

$$\tilde{r} = \begin{pmatrix} \frac{1}{2}\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\lambda\mu}{\lambda^2-\mu^2} & 0 \\ 0 & \frac{\lambda\mu}{\lambda^2-\mu^2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2} \end{pmatrix} . \tag{11}$$

where the above isomorphism (8) is used.

Define the following two quantum R-matrices with spectral paramter:

$$
\begin{aligned}
R &= \mathbf{1} \otimes \mathbf{1} + f(\rho) r \ , \qquad (12)\\
\tilde{R} &= \mathbf{1} \otimes \mathbf{1} + f(\rho) \tilde{r} \ , \qquad (13)
\end{aligned}
$$

where $f(\rho)$ is a smooth function and/or a formal power series.

Lemma 3.1 *R and $\tilde{R}$ fulfill the spectral quantum Yang-Baxter equation, and are unitary up to a factor, i.e.*

$$
\begin{aligned}
R_{12} R_{13} R_{23} &= R_{23} R_{13} R_{12} \ , \qquad (14)\\
R_{21} &\sim R_{12}^{-1}, \qquad (15)
\end{aligned}
$$

and corresponding relations for $\tilde{R}$.

Proof: The proof is a straight forward computation using (9). It only remains to check the vanishing of the terms of third order in $f(\rho)$. The properties for R are well-known, see e.g. [7]. □

We shall call such R-matrices having the properties of the previous lemma quantum R-matrices in short.

4 Quantum integrable Toda like n-particle systems defined by (2×2)-matrices

Let $\mathcal{A}_n$ denote the associative algebra $L(2, \mathbb{K}) \otimes C^\infty(\mathbb{R}^{2n}) \otimes \mathbb{K}(\lambda)$ with tensor factor wise multiplication and pointwise multiplication in $C^\infty(\mathbb{R}^{2n})$. Let $\mathrm{tr} : \mathcal{A}_n \to C^\infty(\mathbb{R}^{2n}) \otimes \mathbb{K}(\lambda)$ denote the standard extension of the matrix trace in the first tensor factor. For any element $U \in \mathcal{A}_1$ let $U^k \in \mathcal{A}_n$, $1 \leq k \leq n$ denote the embedding of $\mathcal{A}_1$ into $\mathcal{A}_n$ by pulling back the matrix elements by means of the projection $p_k : \mathbb{R}^{2n} \to \mathbb{R}^2 : (q, p) \mapsto (q^k, p_k)$.

Given such $U \in \mathcal{A}_2$, we can built the following functions on the phase space $\mathbb{R}^{2n}$ depending on the parameter λ:

Definition 4.1

$$
\begin{aligned}
\chi_n(\lambda) &:= \mathrm{tr}(U^n(\lambda) \ldots U^1(\lambda)), \qquad (16)\\
\tilde{\chi}_n(\lambda) &:= \mathrm{tr}(E_{11} U^n(\lambda) \ldots U^1(\lambda)), \qquad (17)
\end{aligned}
$$

This Definition is motivated by the following important example, the well-known nonperiodic Toda chain whose Hamiltonian function and Lax matrix are given by

$$
\begin{aligned}
H &= \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}, \qquad (18)\\
L &= \sum_{i=1}^{n} p_i \, E_{ii} + \sum_{i=1}^{n-1} e^{\frac{1}{2}(q_i - q_{i+1})} \, (E_{i,i+1} + E_{i+1,i}). \qquad (19)
\end{aligned}
$$

It is easy to see that for the choice $U(q,p)(\lambda) := \begin{pmatrix} -\lambda + p & -e^{-q} \\ e^q & 0 \end{pmatrix}$, the above function $\tilde{\chi}_n(\lambda)$ coincides with the characteristic polynomial $\det(-\lambda \mathbf{1} - L(\vec{q}, -\vec{p}))$. This particular U is taken from

Suris' paper [13] where our U is the transposed matrix of Suris' L and λ is changed to $-\lambda$. The function χ_n is equal to the characteristic polynomial of the periodic Toda chain. It is well-known that the n nonconstant coefficients of these characteristic polynomials are functionally independent and in involution and that the above Hamiltonian function (18) can be obtained from the coefficients of λ^1 and λ^2.

To investigate the quantum case let us define some multiplications:

Definition 4.2 *Define the star product on $\mathcal{A}_n$ as follows*

i.)

$$\begin{array}{rcl} * : \mathcal{A}_n \times \mathcal{A}_n & \to & \mathcal{A}_n, \\ (M * N)_{ik} & := & \sum_j M_{ij} * N_{jk} \end{array}$$

where $M, N \in \mathcal{A}_n$ and $*$ denotes the standard star-product of Weyl type defined in Section 2.

ii.) For $a, b \in L(2,\mathbb{K}) \otimes L(2,\mathbb{K})$ and $f, g \in C^\infty(\mathbb{R}^{2n})$ let $(a \otimes f)(b \otimes g) := ab \otimes fg$.

This definition immediately yields explicit formulae for the star product of two characteristic polynomials.

Proposition 4.3

i.) $\chi_n(\lambda) * \chi_n(\mu) = \mathrm{tr}\big((U_1^n(\lambda) * U_2^n(\mu)) \ldots (U_1^1(\lambda) * U_2^1(\mu))\big)$,

ii.) $\tilde{\chi}_n(\lambda) * \tilde{\chi}_n(\mu) = \mathrm{tr}\big((E_{11} \otimes E_{11})(U_1^n(\lambda) * U_2^n(\mu)) \ldots (U_1^1(\lambda) * U_2^1(\mu))\big)$,

where $U_1 = U \otimes \mathbf{1}$ and $U_2 = \mathbf{1} \otimes U$.

PROOF: This is shown by direct calculation. The λ- and μ-dependent matrices can be reordered without getting additional terms because the order of terms belonging to different pairs of phase space variables is unchanged. $\square$

The main idea to prove quantum commutativity is of course borrowed from the theory of statistical models (see e.g. [7]) and consists in showing that the commutation of $U(\lambda)(q,p)$ and $U(\mu)(q,p)$ as functions of the same phase space variables can be written as a conjugation with special quantum R-matrices. More precisely:

Theorem 4.4 *Let U and R be of the form as shown in the succeding table. Each pair of U and R fullfils the relation*

$$RU_1(\lambda) * U_2(\mu) = U_2(\mu) * U_1(\lambda)R. \tag{20}$$

Lax matrix $U(\lambda)$	quantum R-matrix
$\begin{pmatrix} -\lambda + p & -e^{-q} \\ e^{q} & 0 \end{pmatrix}$ $\begin{pmatrix} -\lambda + p & -e^{-q} \\ pe^{q} & 1 \end{pmatrix}$	$\mathbf{1} \otimes \mathbf{1} + 2\rho r$
$\begin{pmatrix} \frac{1}{\lambda} - \lambda e^{p} & -e^{-q} \\ e^{q} & -\lambda \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\lambda} - \lambda e^{p} & -g^2 e^{-q+p} \\ e^{q} & 0 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\lambda} - \lambda e^{p} & -g^2 (e^{p} + \delta) e^{-q} \\ e^{q} & -\lambda \delta g^2 \end{pmatrix}$	$\mathbf{1} \otimes \mathbf{1} - 2\tanh(\rho)\tilde{r}$
$\begin{pmatrix} \frac{1}{\lambda} - \lambda e^{\epsilon p} & -\epsilon e^{-q} \\ \epsilon e^{q} & -\epsilon^2 \lambda \end{pmatrix}$ $\begin{pmatrix} \frac{1}{\lambda} - \lambda e^{\epsilon p} & -(e^{\epsilon p} - 1) e^{-q} \\ \epsilon e^{q} & +\lambda \epsilon \end{pmatrix}$	$\mathbf{1} \otimes \mathbf{1} - 2\tanh(\epsilon\rho)\tilde{r}$

PROOF: The assertion can directly be proved by a rather long, but straight forward computation. □

The presented Lax matrices U originally occurred in an 'almost complete list' of Toda related classical integrable systems formulated by (2×2)-matrices by Suris in 1997 (cf. [13]). Note that the Lax matrices in this table differ from the ones given by Suris by matrix transposition and the transformation $\lambda \mapsto -\lambda$.

Remark 4.5 *Defining the monodromy matrix* $T(\lambda) := U^n(\lambda) * \cdots * U^1(\lambda)$ *equation* (20) *leads to*

$$RT_1(\lambda) * T_2(\mu) = T_2(\mu) * T_1(\lambda) R, \tag{21}$$

which is the well-known RTT-relation.

As a corollary we obtain the main

Theorem 4.6 *We have the following quantum commutation relations:*

$$\chi_n(\lambda) * \chi_n(\mu) = \chi_n(\mu) * \chi_n(\lambda), \qquad \tilde{\chi}_n(\lambda) * \tilde{\chi}_n(\mu) = \tilde{\chi}_n(\mu) * \tilde{\chi}_n(\lambda)$$

Hence every star polynomial of coefficients of $\chi_n(\lambda)$ *or* $\tilde{\chi}_n(\lambda)$ *defines a quantum integrable system. In particular, the nonperiodic and periodic Toda chains are quantum integrable.*

In general we can consider two sets of classical conserved quantities: Given a Lax matrix, one takes the trace polynomials $\{I_1, \ldots, I_n\}$, where $I_k = (1/k)\mathrm{tr} L^k$, or the coefficients of the characteristic polynomial $\{J_1, \ldots, J_n\}$, where we use the notation

$$\chi_n(\lambda) = \det(\lambda \mathbf{1} - L) = \lambda^n + \sum_{k=1}^{n} J_k \lambda^{n-k}. \tag{22}$$

Lemma 4.7 *Let $r = (r_1, \ldots, r_n)$ be a multiindex ($r! := r_1! \ldots r_n!$, $|r| := r_1 + \cdots + r_n$) and $\alpha = (1, 2, \ldots, n)$. The relation between classical trace polynomials and coefficients of the characteristic polynomial is expressed by the Waring's formulae (see e.g. [2]):*

$$J_k = \sum_{\alpha r = k} \frac{(-1)^{|r|}}{r!} I_1^{r_1} \ldots I_k^{r_k}, \tag{23}$$

$$I_k = \sum_{\alpha r = k} \frac{(-1)^{|r|}}{r!} \frac{|r|!}{|r|} J_1^{r_1} \ldots J_k^{r_k}. \tag{24}$$

Classical, the involutivity of one set follows from the involutivity of the other set. Quantum mechanical it does not, but, since we have shown that the J_k ($1 \leq k \leq n$) are in involution, we know that the star commutator of the corresponding star polynomials of (24), say $\hat{I}_k$, which one gets by replacing the usual multiplication by star products, vanishes. In the lowest order in $\hbar$ $\hat{I}_k$ coincides with I_k. Therefore we can calculate quantum corrections for the classical trace polynomials of the Toda chain with Lax matrix (19): For $k \leq 3$ $\hat{I}_k$ and I_k are the same, for greater k one gets:

$$\hat{I}_4 = I_4 + \rho^2 \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}, \tag{25}$$

$$\hat{I}_5 = I_5 + 2\rho^2 \sum_{i=1}^{n-1} (p_i + p_{i+1}) e^{q_i - q_{i+1}}, \tag{26}$$

$$\begin{aligned} \hat{I}_6 = {} & I_6 + \rho^2 \sum_{i=1}^{n-1} \left(\frac{8}{3} e^{2(q_i - q_{i+1})} + \frac{10}{3} (p_i^2 + p_i p_{i+1} + p_{i+1}^2) e^{q_i - q_{i+1}} \right) \\ & + \rho^2 \sum_{i=1}^{n-2} e^{q_i - q_{i+2}} + \rho^4 \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}. \end{aligned} \tag{27}$$

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References

[1] Agarwal, G.S., Wolf, E.: *Calculus for Functions of Noncommuting Operators and General Phase Space Methods in Quantum Mechanics. I. Mapping Theorems and Ordering of Functions on Noncommuting Operators.* Phys.Rev. D **2** 10 (1970), 2161-2188.

[2] Arnaudiès, J. M., Fraysse, H.: *Cours de mathématiques I, Algèbre*, Dunod, Paris, 1992.

[3] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and quantization*, Ann.Phys. **111** (1978), 61-110.

[4] Bordemann, M.: *Generalized Lax pairs, the modified classical Yang-Baxter equation, and affine geometry of Lie groups*, Comm. Math. Phys., 135 (201-216), 1990.

[5] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov Star Products on Cotangent Bundles I: Weyl and Standard Ordering with Differential Operator Representation*, Preprint Uni Freiburg FR-THEP 97/10, q-alg/9707030, to appear in Commun. Math. Phys.

[6] Calogero, P., Ragnisco, O., Marchioro, C.: *Exact solutions of the classical and quantal one-dimensional many-body problems with the two-body potential $V_a(x) = g^2a^2/sinh^2(ax)$*, Lett. Nuovo Cimento **13** (1975), 383-387.

[7] Chari, V., Pressley, A.: *Quantum groups*, Cambridge University Press, 1994.

[8] DeWilde, M., Lecomte, P.: *Existence of star-products and of formal deformations of Poisson-Lie algebra of arbitrary symplectic manifolds*, Lett.Math.Phys. **7** (1983), 487-496.

[9] Faddeev, L.D., Takhtajan, L.A.: *Hamiltonian methods in the theory of solitons*, Springer, 1986.

[10] Hikami, K., Ujino, H., Wadati, M.: *Integrability of the Quantum Calogero-Moser model*, J.Phys.Soc. Japan **61** (1992), 3425-3427.

[11] Olshanetzky, M.A., Perelomov, A.M.: *Quantum integrable systems related to Lie algebras*, Phys.Rep **94** (1983), 313-404.

[12] Perelomov, A.M.: *Integrable systems of classical mechanics and Lie algebras*, Birkhäuser, Boston, 1990.

[13] Suris, Y.B.: *A collection of integrable systems of the Toda type in continous and discrete time, with 2×2 Lax representations*, Preprint solv-int/9703004v2, 1997.

[14] Walter, M.: *Zwei Familien integrabler Systeme in der klassischen Mechanik*, Diploma thesis, University of Freiburg, March 1998.